BOLT BERANEK AND NEWMAN INC

CONSULTING · DEVELOPMENT · APPLIED RESEARCH



Report No. 774
Job No. 110022

# COMPENDIUM OF IMPEDANCE FORMULAS

by Manfred A. Heckl

Contract Nonr 2322(00)
Task No. NR 264-017



26 May 1961

Submitted to:

Office of Naval Research
Code 411
Washington 25, D. C.

Attention: Mr. Marvin Lasky



CAMBRIDGE, MASSACHUSETTS CHICAGO, ILLINOIS LOS ANGELES, CALIFORNIA

# COMPENDIUM OF IMPEDANCE FORMULAS

by

Manfred A. Heckl

Contract Nonr 2322(00)
Task No. NR 264-017
BBN Report No. 774
BBN Job No. 110022

26 May 1961

Submitted to:

Office of Naval Research
Code 411
Washington 25, D. C.

Attention: Mr. Marvin Lasky

BOLT BERANEK AND NEWMAN INC.
50 Moulton Street
Cambridge 38, Massachusetts

# COMPENDIUM OF IMPEDANCE FORMULAS

Table of Contents Page

Table of Contents (continued) Page

# COMPENDIUM OF IMPEDANCE FORMULAS

## ABSTRACT

A list of impedance formulas is presented which can be applied to the vibration excitation of simple structures such as beams, plates, rings, and beam-plate systems. Infinite, semi-infinite, and finite systems are considered. Most of the formulas have been taken from the existing literature; others are derived. In the latter case short derivations are given in the Appendixes.

## LIST OF SYMBOLS

$a$ = radius

$b$ = width of a beam

$c$ = propagation velocity for bending waves

$c_L$ = propagation velocity for longitudinal waves

$c_T$ = propagation velocity for torsional waves

$|d|^2$ = transmission coefficient

$f$ = frequency

$h$ = thickness

$i$ = $\sqrt{-1}$

$k$ = $\frac{\omega}{c}$ = wave number for bending waves

$\ell$ = length

$m$ = surface mass or linear mass

$r$ = reflection coefficient

$v$ = particle velocity

$w$ = angular velocity

$x, y$ = coordinates

$B$ = bending stiffness of beams; for rectangular cross section

$$B = \frac{Eh^3b}{12}$$

$B'$ = torsional stiffness of beams

$D$ = bending rigidity of plates; for rectangular cross section

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

| | | |
|---|---|---|
| $D'$ | = | torsional rigidity of plates |
| $E$ | = | Young's modulus |
| $F$ | = | force |
| $G$ | = | shear stiffness |
| $L$ | = | length |
| $M$ | = | moment |
| $P$ | = | mechanical power |
| $R$ | = | reflection coefficient for bending wave near fields |
| $S$ | = | cross sectional area of beams |
| $Z_F$ | = | force impedance |
| $Z_M$ | = | moment impedance |
| $Z_S$ | = | source impedance |
| $\eta$ | = | loss factor |
| $\kappa$ | = | radius of gyration |
| $\lambda$ | = | wavelength |
| $\mu$ | = | Poisson's number |
| $\rho$ | = | density |
| $\omega = 2\pi f$ | = | angular frequency |

# I. INTRODUCTION

The concept of mechanical impedance is very useful in solving vibration problems because it allows expression of the energy transfer from a vibration source to a structure, and from one structure to another, in fairly simple terms. It is an especially convenient concept for engineers with background in electrical engineering because of the analogy with electrical and mechanical impedances.

A particular example of the usefulness of the impedance concept is the design of vibration mounts, where the vibration reduction can be predicted rather accurately if all the impedances involved are known. But there is one difficulty: In many practical cases the impedances are not known, and therefore one is forced to represent certain parts of structures by lumped masses, springs and dashpots. This approach is convenient and useful for low frequencies or for heavy and very compact structures, but for higher frequencies and light structures it may be very misleading.[1-4] Thus, one is faced with the problem of getting more information about the impedances of real structures.

It is hoped that the present report will provide some of this information. The report gives the impedance of structures which are more complicated than lumped masses and springs, but still simple enough to be treated mathematically.

Most of the formulas of this report deal with infinite systems. This is not as serious a restriction as it might appear provided that one is not interested in the response of a system at a given frequency, but rather in the average behavior in a frequency band.

It will even be shown in Section II-A4 that the input impedance of an infinite system governs the flow of mechanical power into a finite system of the same kind.

There are two types of impedances listed in this report; one of them is the force impedance $Z_F$ which is given by

$$Z_F = \frac{F}{v} \tag{I-1}$$

($F$ = exciting force, $v$ = particle velocity at the excitation point); the other one is the moment impedance $Z_M$ given by

$$Z_M = \frac{M}{w} \tag{I-2}$$

($M$ = exciting moment, $w$ = angular velocity at the excitation point).

In both cases only point forces or moments are considered. This means that the excitation is localized in a region which is very small compared to the wavelength in the structure, otherwise the formulas cannot be applied.

The more familiar force impedance is used if all the excitation is given by a force; one example might be the excitation of a structure by an engine (pump, etc.) provided that the vibrations of the engine (etc.) are purely perpendicular to the structure. If there is also some "rocking" of the engine, the exciting moment, and therefore the moment impedance, has to be taken into account, too.

Since most of the formulas given in this report have restricted ranges of applicability, an attempt has been made to give not only the equations but also the corresponding restrictions. With regard to sign convention we emphasize that we have assumed harmonic motion of angular frequency $\omega$ with the time dependence expressed by $e^{i\omega t}$. This is in agreement with Cremer's work[5] from which many of the formulas were taken. If harmonic motion of the form $e^{-i\omega t}$ is assumed, $-i$ instead of $+i$ must be inserted in the formulas. Thus, in the present report an impedance with a positive imaginary part is mass-like, while an impedance with a negative imaginary part is stiffness-like.

## II. FORCE IMPEDANCE OF BEAMS

### A. Excitation of Bending Waves

The impedance of beams or systems of beams (for sketches of the cases considered in the following, see Fig. 1) which are excited by a point force acting perpendicularly to the beam can be calculated by solving the one-dimensional bending wave equation

$$\frac{d^4v}{dx^4} - k^4v = 0 \qquad \text{(II-1)}$$

under the proper boundary conditions. In Eq. (II-1), v is the transverse velocity of the beam

$$k = \left(\omega^2 \frac{m}{ES\kappa^2}\right)^{1/4} \qquad \text{(II-2)}$$

the bending wave number, m the mass per length, S the cross-sectional area of the beam, $\kappa$ the radius of gyration, and E is Young's modulus.

Equation (II-1) is valid, provided that the bending wavelength $\lambda = 2\pi/k$ is much larger than the thickness h of the beam. In general, Eq. (II-1) can be used if

$$\lambda \geq 6h \quad . \qquad \text{(II-3)}$$

A similar, somewhat less stringent restriction holds for the width of the beam.

1. Impedance of Uniform Infinite Beams

The term "infinite beam" does not mean that the beam must actually be of infinite length. All the following results are correct if the beam is highly damped or otherwise terminated in such a manner so that no reflected bending waves come back from that side of the beam which is assumed to go to infinity. Methods for terminating beams have been described by Kurtze, Tamm, and Vogel.[6]

a) Beam extending from x = -∞ to x = +∞

In this case (investigated by Cremer[5] the boundary conditions at the driving point are, first that the angular velocity is zero, and second, that the exciting force is equal to the transverse force of the bending wave. This gives for the impedance

$$Z_F = 2mc(1+i) = \frac{2SE\kappa^2 k^3}{\omega}(1+i) \qquad \text{(II-4)}$$

where $c = \frac{\omega}{k}$ = bending wave velocity. The geometry for this case is shown in Fig. 1a.

b) Beam extending from x = 0 to x = ∞

This case was also considered by Cremer.[5] See Fig. 1b for a sketch of the geometry. Using as boundary conditions, first that the exciting force is equal to the transverse force of the bending wave and second, that there is no bending moment, he obtained

$$Z_F = \frac{mc}{2}(1+i) \qquad \text{(II-5)}$$

## 2. Impedance of Beams with One Reflecting Device

To make the following formulas as general as possible we define a reflecting device as any inhomogeneity in material or cross-section which causes a reflection of bending waves. A list of reflection coefficients r for some special cases is given in Appendix A. This list includes the reflection coefficients for the reflected non-propagating wave (near field) as well as the reflected propagating wave (far field). We shall always assume that the distance between the driving force and the closest reflecting device is greater than half a wavelength. This allows us to neglect the complications caused by the reflected non-propagating wave (near field).

### a) Beams extending from $x = -\infty$ to $x = +\infty$ with a reflecting device at $x = -\ell$; ($\ell$ may be positive or negative)

The impedance of this structure (shown in Fig. 1c) can be found as a limiting case of Eq. (B6) in Appendix B. This equation gives for $r_\ell = r$, $r_L = 0$

$$Z_F = \frac{4mc}{1-i + re^{-2ik\ell}} \tag{II-6}$$

### b) Beams extending from $x = 0$ to $x = +\infty$ with a reflecting device at $x = \ell$

In this case (shown in Fig. 1d) a derivation very similar to the one in Appendix B can be made. The main difference is that the impedance of the initial wave is given by Eq. (II-5) instead of Eq. (II-4). Furthermore, use is made of the fact that for a free end (at $x = 0$) the reflection coefficients are $-i$ and $1-i$, respectively. Thus, we get the following expression for the impedance

$$Z_F = \frac{mc}{2}(1+i)\frac{1 + ire^{-2ik\ell}}{1 + re^{-2ik\ell}} \quad . \qquad \text{(II-7)}$$

For many applications it might be of interest to find the mean square velocity $\overline{v_o^2}$ at the driving point if the beam is excited by a broad frequency band. Contour integration of Eq. (II-7) shows that

$$\overline{v_o^2} \approx \frac{2|F|^2}{m^2c^2}\frac{1 + |r|^2}{1 - |r|^2} \quad . \qquad \text{(II-8)}$$

This result can be extended easily to beams with internal losses characterized by a loss factor $\eta$. In this case we have

$$\overline{v_o^2} \approx \frac{2|F|^2}{c^2m^2}\frac{1 + |r|^2\, e^{-\eta k\ell}}{1 - |r|^2\, e^{-\eta k\ell}} \quad . \qquad \text{(II-9)}$$

3. Impedance of Beams with More Reflecting Devices

a) Beams extending from $x = -\infty$ to $x = +\infty$, with reflecting devices at $x = -L$ and $x = \ell$ ($L$ and $\ell$ positive)

This case, shown in Fig. 1e, is treated in the second part of Appendix B. Eq. (B6) derived therein leads to the following expression for the force impedance

$$Z_F = 4mc\,\frac{1-r_\ell r_L e^{-2ik(\ell+L)}}{1-i+r_L e^{-2ikL}+r_\ell e^{-2ik\ell}+(1+i)r_\ell r_L e^{-2ik(\ell+L)}} \qquad \text{(II-10)}$$

(For $r_\ell = 0$, Eq. (II-10) is identical with Eq. (II-6)).

The mean square velocity at the driving point for broadband excitation is obtained by integrating Eq. (B6) over frequency. This gives

$$\overline{v_o^2} \approx \frac{|F|^2}{16c^2m^2} \frac{2 + |r_L|^2 + |r_\ell|^2}{1 - |r_L r_\ell|^2} \quad . \tag{II-11}$$

If the damping of the beam is significant, Eq. (II-11) can be extended by inserting $|r_\ell r_L|^2 \, e^{-\eta k(\ell+L)}$ in place of $|r_\ell r_L|^2$.

b) <u>Beams extending from $x = -\infty$ to $x = +\infty$, with several reflecting devices on both sides of the driving point.</u>

Equation (II-10) also holds for this case. We only have to assume that instead of simple reflecting devices at $x = -L$ and $x' = \ell$, there are two "black boxes" containing more than one inhomogeneity. The problem is now reduced to determining the reflection coefficients $r_1$ and $r_2$ of the "black boxes." Unfortunately the reflection coefficients of systems more complicated than those given in Appendix A are hard to compute. But the average behavior, which is given by Eq. (II-11), can easily be found if we remember that (if no internal losses occur) the energy coming to a reflecting point is equal to the sum of the reflected and transmitted energy. Or in terms of the reflection coefficient $|r|$ and the transmission coefficient d

$$|r|^2 = 1 - |d|^2 \quad . \tag{II-12}$$

If we insert this equation in (II-11), we get

$$\frac{\overline{v_o^2}}{|F|^2} \approx \frac{1}{16c^2m^2} \frac{3 - |d_1|^2 - |d_2|^2}{|d_1^2|+|d_2^2|-|d_1 d_2|^2} \approx \frac{1}{16c^2m^2} \frac{3}{|d_1^2|+|d_2^2|} \quad . \qquad \text{(II-13)}$$

The latter approximation holds for $|d_1| \ll 1$, $|d_2| \ll 1$.

Equation (II-13) is correct if the damping is small. This condition can be written as:

$$|d_1^2| + |d_2|^2 > e^{-\eta k(\ell+L)} \quad .$$

4. <u>The Power Flow Into Finite Beams</u>

At this stage it seems appropriate to give a general expression for the average mechanical power that flows into a beam from a vibration source. To this end we consider a finite beam of length $\ell$ and loss factor $\eta$. This beam is excited by a point force F at the point $\ell_o$. The velocity of the beam can then be found in terms of the eigenfunctions $\phi_n(x)$ and eigenfrequencies $\omega_n$. Thus we get (for example see Ref. 7)

$$v = \frac{F}{m} \sum_{n=0}^{\infty} \frac{i\omega\phi_n(\ell_o)}{\omega_n^2 - \omega^2 + i\eta\omega_n^2} \frac{\phi_n(x)}{\int \phi_n^2 \, dx} \quad . \qquad \text{(II-14)}$$

Therefore the mechanical power P transmitted into the beam is

$$P = \frac{1}{2} \operatorname{Re} \left\{ F v_o \right\} = \frac{F^2}{2m} \sum_{n=0}^{\infty} \frac{\omega\eta\omega_n^2}{|\omega_n^2 - \omega^2 + i\eta\omega_n^2|^2} \frac{\phi_n^2(\ell_o)}{\int \phi_n^2 \, dx} . \qquad \text{(II-15)}$$

Averaging over all possible exciting points $\ell_o$ and over a broad frequency band $\Delta\omega$ gives

$$\overline{P} = \frac{\pi}{4} \frac{F^2}{m\ell} \frac{\Delta N}{\Delta\omega} \quad . \tag{II-16}$$

Here $\Delta N$ is the number of resonance frequencies in the frequency band $\Delta\omega$. If there are several resonance frequencies within the band of interest, we can approximate Eq. (II-16) by inserting the asymptotic formula

$$\frac{\Delta N}{\Delta\omega} \approx \frac{c\ell}{2\pi} \quad . \tag{II-17}$$

Thus we get

$$\overline{P} \approx F^2 \frac{1}{8cm} = \frac{1}{2} F^2 \, \mathrm{Re}\left\{\frac{1}{Z_\infty}\right\} \tag{II-18}$$

which is exactly the formula that gives the power that is transmitted from a point source into an infinite beam (see Eq. (II-4)).

Equations (II-16) and (II-18) show that the average power is independent of the loss factor $\eta$ and the length of the beam. This fact has an interesting consequence for the power transmitted into a beam with reflecting devices.

If the reflection coefficients are rather high it is a good approximation to assume that the part of the beam which is driven and which is between two reflecting devices (which are a distance $\ell$ apart)

behaves like a finite beam of length $\ell$ with additional damping. This means that Eqs. (II-16) and (II-18) hold in this case, too; or to put it in another way, the average mechanical power which is transmitted into a beam is, approximately, unaffected by the presence of reflecting devices, and can easily be computed if the impedance of the corresponding infinite system is known.

5. <u>Impedance of Beams Driven with Non-Zero Impedance Sources</u>

In all formulas given above we have assumed that the source does not influence the beam impedance; i.e., the source impedance $Z_s$ has been assumed zero, or at least much smaller than the impedance of the beam. But there are many cases in practice where the source impedance $Z_s$ cannot be neglected (e.g., a heavy machine mounted on a beam).

For the cases Sec. (II-A1) (without reflecting devices) the influence of the source can easily be computed by adding the source impedance to the beam impedance as given in Eqs. (II-4) or (II-5).

For all other cases (with reflecting devices) simply adding the source impedance would be incorrect, because of the interaction at the driving point of the source and the waves reflected back from the reflecting devices. We therefore must know the reflection coefficients for the propagating waves and non-propagating waves at the driving point. If these quantities are denoted $r_s$ and $R_s$ respectively, we get by extension of Section II.A.2.a

$$Z_F = \left[2mc(1+i) + Z_s\right] \frac{(1-i)\left(1-rr_s\ e^{-2ik\ell}\right)}{1-i+re^{-2ik\ell}(1+R_s+ir_s)} \qquad \text{(II-20)}$$

Extension of Eq. (II-18) in Section II.A.2.b gives

$$Z_F = \left[\frac{mc(1+i)}{2} + Z_s\right] \frac{1 - r_s \, re^{-2ik\ell}}{1 + \frac{1}{2} \, re^{-2ik\ell}(1+R_s-r_s)} \quad . \qquad \text{(II-21)}$$

We see that Eqs. (II-20) and (II-21) cannot be obtained by adding $Z_s$ to Eq. (II-6) or Eq. (II-7). The reason for this somewhat surprising result is that in general the reflection coefficient $r_s$ at the source does not depend only on the force impedance $Z_s$ of the source but also on the moment impedance at the driving point. It can be shown for example that Eq. (II-21) becomes

$$Z = \frac{mc}{2}(1+i) \, \frac{1+ire^{-2ik\ell}}{1+re^{-2ik\ell}} + Z_s \qquad \text{(II-22)}$$

(compare with Eq. II-7) provided that the moment impedance of the source vanishes. Equation (II-22) can be proved by putting $v = 0$ in Eqs.(A13) and (A14) and introducing the resulting expressions into Eq. (II-21).

B. Excitation of Longitudinal Waves

The impedance of beams (for sketches of the geometries see Fig. 2) excited by a force acting in the direction of the beam axis can be computed from the equation for longitudinal vibrations in beams

$$\frac{d^2v}{dx^2} + \frac{\omega^2}{c_L^2} v = 0 \quad . \qquad \text{(II-23)}$$

In this equation v is the longitudinal particle velocity,

$$c_L = \sqrt{\frac{E}{\rho}} \tag{II-24}$$

is the velocity of propagation for longitudinal waves, E is Young's modulus, and $\rho$ is the density of the beam material.

Equations (II-23) and (II-24) are valid only when the cross-sectional dimensions of the beam are very small compared to the longitudinal wavelength.

## 1. Impedance of Uniform Infinite Beams

The longitudinal impedance of a beam extending from $x = -\infty$ to $x = +\infty$ and driven at some point in between is given by (see Fig. 2a)

$$Z_F = 2S\ \rho c_L = 2S\sqrt{E\rho} \tag{II-25}$$

(S = cross section of the beam).

For a beam extending from $x = 0$ to $x = +\infty$ and driven at $x = 0$ the impedance is (see Fig. 2b)

$$Z_F = S\rho c_L = S\sqrt{E\rho} \tag{II-26}$$

## 2. Impedance of Beams with Reflecting Devices

The impedance of beams with discontinuities can be calculated by using the methods given in Appendix B. Since all waves are propagating, the formulas obtained are correct even if the distance between the source and the discontinuity is much smaller than a wavelength. Furthermore there is a simple relation between the reflection coefficient and the impedance looking into the discontinuity; thus the behavior of the beam can be expressed solely by impedances in a fairly simple way.

a) <u>Beams extending from $x = 0$ to $x = \infty$ with a reflecting device at $x = \ell$ $(\ell > 0)$</u>

If the reflection coefficient at $x = \ell$ is r we get (see Fig. 2c)

$$Z_F = S\sqrt{E\rho}\ \frac{1-re^{-2ik\ell}}{1+re^{-2ik\ell}} \tag{II-27}$$

$\left(k = \frac{\omega}{c_L}\right)$.

If $Z_D$ is the impedance looking into the discontinuity the reflection coefficient can be written as

$$r = \frac{Z_D - S\sqrt{E\rho}}{Z_D + S\sqrt{E\rho}} \tag{II-28}$$

b) <u>Beams extending from $x = -\infty$ to $x = +\infty$ with reflecting devices at $x = \ell$ and $x = -L$</u>

In this case the impedance turns out to be (see Fig. 2d)

$$Z_F = 2S\sqrt{E\rho}\ \frac{1 - r_\ell r_L\, e^{-2ik(\ell+L)}}{1 + r_\ell e^{-2ik\ell} + r_L e^{-2ikL} + r_\ell r_L e^{-2ik(\ell+L)}} \tag{II-29}$$

($r_\ell$ = reflection coefficient at $x = \ell$; $r_L$ = reflection coefficient at $x = -L$).

The values for $r_\ell$ and $r_L$ can be obtained from Eq. (II-28) if the impedances at $x = \ell$ and $x = -L$ are known.

Finally, it should be mentioned that for longitudinal waves the effect of a finite source impedance $Z_S$ can always be taken into account simply by adding it to the impedance of the structure.

## III. FORCE IMPEDANCE OF PLATES

### A. Impedance of Infinite Isotropic Plates

If a force F acts perpendicularly to a plate, the impedance can be found by solving the plate equation which is an extension of Eq. (II-1). If $\Delta$ is the Laplacian operator, the velocity v of the plate is given by solutions of

$$\Delta\Delta v - k^4 v = 0 \quad . \tag{III-1}$$

As in the one-dimensional case, this equation - and therefore the impedance given below - is only correct when inequality Eq. (II-3) is fulfilled.

The impedance of point driven plates has been calculated by several authors[8,9,10] in different ways. The result is

$$Z_F = \frac{8\omega m}{k^2} = 8\sqrt{Dm} \quad . \tag{III-2}$$

In this equation D is the bending rigidity* and m the surface mass of the plate.

Equation (III-2) also holds for plates consisting of several layers, provided that they are connected in such a way that only pure bending motion occurs. In this case D is the bending rigidity of the combination and m its surface mass. For homogeneous plates Eq. (III-2) can also be expressed in terms of Young's modulus E or in terms of the velocity of longitudinal waves $c_L$. Thus we get

---

* Because of the lateral contraction, the bending rigidity of beams is always a little smaller than the bending rigidity of plates. But for most practical cases they can be set equal

$$Z_F = 2.3\, h^2 \sqrt{\frac{E\rho}{1-\mu^2}} = 2.3\, \frac{c_L\, \rho h^2}{1-\mu^2} \approx 2.3\, c_L\, \rho h^2 \qquad \text{(III-2a)}$$

($\rho$ = density, $\mu$ = Poisson's ratio.)

## B. Impedance of Infinite Orthotropic Plates

Orthotropic plates have different bending stiffnesses for vibrations in different directions. Examples of orthotropic plates are: plates made of non-isotropic material, plates with grooves or ribs, corrugated plates, or grillages consisting of crossed beams.

If the distance between the grooves, ribs, etc. (whatever is larger) and the plate thickness is much smaller than the shortest bending wavelength on the plate, the bending vibrations are described by the following equation first obtained by Huber[12]:

$$D_x \frac{\partial^4 v}{\partial x^4} + 2D_{xy} \frac{\partial^4 v}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 v}{\partial y^4} - \omega^2 m v = 0 \quad . \qquad \text{(III-3)}$$

Here $D_x$ is the bending rigidity in the stiffest direction (i.e., for example, in the direction of the grooves or ribs), $D_y$ is the rigidity in the least stiff direction (i.e., perpendicular to the grooves or ribs), and m is the average surface mass. $D_{xy}$ is also a kind of rigidity which may either be measured (see the papers by Hoppman[13]) or calculated.

For a corrugated plate (illustrated in Fig. 3a) the rigidities are given by[14,15]

$$D_x = EI_x \quad ; \quad D_y = \frac{g}{s}\,\frac{Eh^3}{12(1-\mu^2)} \quad ; \quad D_{xy} = \frac{s}{g}\,\frac{Eh^3}{12(1+\mu)} \quad ; \qquad \text{(III-4)}$$

for a plate with ribs (illustrated in Fig. 3b) we have

$$D_x = EI_x \quad ; \quad D_y = \frac{Euh^3}{12\left[u+t\left(\frac{h^3}{H^3}-1\right)\right]} \quad ; \quad D_{xy} = 2D' + \frac{B'}{u} \,. \qquad \text{(III-5)}$$

In these equations E is Young's modulus, $I_x$ is the moment of inertia along the x axis, $\mu$ is Poisson's ratio, D' is the torsional rigidity of the plate without ribs, and B' is the torsional stiffness of one rib. The dimensions g,s,h,u,t,H are defined in the figures.

Finally for grills consisting of perpendicular beams (illustrated in Fig. 3c) we get

$$D_x = \frac{B_1}{\ell_1} \quad ; \quad D_y = \frac{B_2}{\ell_2} \quad ; \quad D_{xy} = \frac{1}{2}\left(\frac{B_1'}{\ell_1} + \frac{B_2'}{\ell_2}\right) \,. \qquad \text{(III-6)}$$

Here B is the bending stiffness and B' the torsional stiffness of the beams (see Fig. 3).

The force impedance of orthotropic plates for bending waves was recently calculated by Heckl.[16] The result is

$$Z_F = 4\pi\,\frac{(m^2 D_x D_y)^{1/4}}{K(a)} \qquad \text{(III-7)}$$

with $a = \frac{1}{\sqrt{2}}\left(1 - \frac{D_{xy}}{\sqrt{D_x D_y}}\right)^{1/2}$

In this equation K(a) is the complete elliptical integral of the first kind, for which tabulated values are available (for example see Ref. 17).

If $D_{xy} > \sqrt{D_x D_y}$ the values of a become imaginary. In this case the transformation

$$K(i\alpha) = \frac{1}{\sqrt{1+\alpha^2}} \quad K\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)$$

for $a = i\alpha$ has to be made.

For most practical cases $D_{xy} \approx \sqrt{D_x D_y}$, then Eq. (III-7) can be approximated by

$$Z_F \approx 8 \left(m^2 D_x D_y\right)^{1/4} . \qquad \text{(III-8)}$$

Comparison with Eq. (III-2) shows that for orthotropic plates $\sqrt{D_x D_y}$ is analogous to D in isotropic plates.

C. Impedance of a Homogeneous Plate, Infinite in One Direction, Simply Supported in the Other One

In this case the impedance can be found by expanding the velocity and the force in series of the form $\sum_{n=1}^{\infty} f(x) \sin \frac{n\pi y}{\ell}$ and inserting these in the bending wave equation. After some calculations the impedance turns out to be given by

$$\frac{1}{Z_F} = \frac{4i}{8\sqrt{Dm}} \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n^2\pi^2-k^2\ell^2}} - \frac{1}{\sqrt{n^2\pi^2+k^2\ell^2}}\right)\sin^2 \frac{n\pi y_0}{\ell} \qquad \text{(III-9)}$$

($\ell$ is the distance between the support edges, $y_o$ the coordinate of the driving point). The geometry is shown in Fig. 4.

As pointed out in Eq. (C-20) of Appendix C, Eq. (III-9) can be obtained also as a limiting case of the impedance of a beam-plate system.

For very small values of $\ell$ and low frequencies: i.e., for $k\ell < \frac{\pi}{2}$, Eq. (III-9) can be approximated by

$$\frac{1}{Z_F} \approx \frac{i\omega\ell^2}{2\pi^3 D} \sin^2 \frac{n\pi y_o}{\ell} \tag{III-9a}$$

We see that in this case the plate acts as a spring with a stiffness constant given by

$$s = \frac{2\pi^3 D}{\ell^2 \sin^2 \frac{n\pi y_o}{\ell}} \quad .$$

For very large values of $\ell$ and high frequencies, i.e., for $k\ell \gg \pi$ the sums in Eq. (III-9) can be evaluated to give $Z_F = 8\sqrt{Dm}$ which is the impedance of a plate infinite in both directions.

For the intermediate range, the reciprocal impedance for a plate driven in the center ($y_o = \frac{\ell}{2}$) is plotted in Fig. 4 and Fig. 5. The full lines show the real part, the dashed lines the imaginary part. It can be seen that for $n > 9$ g the average value of $\mathrm{Re}\left\{\frac{8\sqrt{Dm}}{Z_F}\right\}$ is very close to unity, and the average value of $\mathrm{Im}\left\{\frac{8\sqrt{Dm}}{Z_F}\right\}$ is nearly zero. Thus for high frequencies the impedance of such a semi-infinite plate is very close to the impedance of an infinite plate [see Eq. (III-2)]. This result also holds for plates finite

in both directions (see, e.g., Skudrzyk[18]). If one is especially interested in the mechanical power transmitted by a point force into a finite isotropic or orthotropic plate, one can show that the power averaged in frequency is inversely proportional to the force impedance of the infinite system. This fact can be expressed by the formula

$$\overline{P} = \frac{1}{2}|F|^2 \operatorname{Re}\left\{\frac{1}{Z_\infty}\right\} \qquad \text{(III-10)}$$

The proof for this equation can be given in exactly the same way as it was given in the one-dimensional case (see Section II-A4).

## IV. FORCE IMPEDANCE OF BEAM-PLATE SYSTEMS

### A. Impedance of Systems Infinite in Both Directions

This problem (see Fig. 6a) was solved by G. Lamb[19] using the notations

$m_B$ = mass per length of the beam,

$m_P$ = surface mass of the plate,

B = bending stiffness of the beam,

D = bending rigidity of the plate,*

$$k_B = \frac{\omega}{c_B} = \left(\omega^2 \frac{m_B}{B}\right)^{1/4} \; ; \; k_P = \frac{\omega}{c_P} = \left(\omega^2 \frac{m_P}{D}\right)^{1/4} \; ; \; r = \frac{c_P}{c_B} = \frac{k_B}{k_P} \; ,$$

the result is**

$$\frac{1}{Z_F} = \frac{ik_B^3}{2\pi m_B c_B} \int_{-\infty}^{+\infty} \frac{dx}{x^4 - k_B^4 + \frac{2iD}{B}(k_P^4 - x^4)\left[(k_P^2 - x^2)^{-1/2} + i(k_P^2 + x^2)^{-1/2}\right]} \qquad \text{(IV-1)}$$

---

* It should be noted that $m_B$ and $m_P$, and B and D have different dimensions, e.g., in the case of rectangular cross-sections we have

$$m_B = \rho_B h_B b, \; m_P = \rho_P h_P \quad B = \frac{E_B h_B^3 b}{12} \; ; \; D = \frac{E_P h_P^3}{12(1-\mu^2)}$$

For dimensions see Fig. 6a.

** Since we assumed harmonic motion of the form $e^{+i\omega t}$ we have +i where there is -i in Lamb's expressions.

For $k_B \frac{B}{D} r^3 = k_P \frac{m_B}{m_P} \gg 1$ the following approximation can be obtained

$$\frac{1}{Z_F} \approx \frac{1}{4m_B c_B}\left[1-i+\frac{4}{\pi}\frac{iD}{k_B B}\left(1+\frac{3-r^2}{2r^3(1-r^2)^{1/2}}\cos^{-1}r+\frac{3+r^2}{2r^3(1+r^2)^{1/2}}\sinh^{-1}r\right)\right]. \quad \text{(IV-2)}$$

Physically Eq. (IV-2) shows that if the beam is much stiffer than the plate and the frequency not too low, the force impedance of a beam-plate system is nearly the same as the impedance of the beam alone [compare with Eq. (II-4)].

On the other hand it can be shown that if the stiffness of the beam is not much higher than that of the plate, the impedance of the beam-plate system is approximately the impedance of the plate given by Eq. (III-2).

B. Impedance of Systems, Infinite in One Direction and Simply Supported in the Other One

This problem, whose geometry is shown in Fig. 6b, is treated in Appendix C. Basically, the calculations show that for beams much stiffer than the plate, the beam-plate system behaves like a simply supported beam with a loss factor

$$\eta \approx \frac{2}{k_P}\frac{m_P}{m_B} \quad \text{(IV-3)}$$

and an additional linear mass

$$\Delta m_B = \frac{2}{k_P} m_P \tag{IV-4}$$

The added damping and mass arise from the waves coupled to the plate from the beam.

## V. FORCE IMPEDANCE OF RINGS

According to Love[20] the radial velocity v of a ring with radius a, linear mass m and bending stiffness B is given by

$$\frac{B}{ma^4}\left(\frac{\partial^6 v}{\partial\phi^6} + 2\,\frac{\partial^4 v}{\partial\phi^4} + \frac{\partial^2 v}{\partial\phi^2}\right) + \omega^2 v - \omega^2\,\frac{\partial^2 v}{\partial\phi^2} = \frac{i\omega}{m}\left(\frac{\partial^2 p_r}{\partial\phi^2} - \frac{\partial p_t}{\partial\phi^2}\right) . \qquad \text{(V-1)}$$

In this equation $p_t$ is the pressure in the tangential direction, which in our case is zero, $p_r$ is the pressure in radial direction, which in our case is $F\delta(\phi)$ ($F$ = driving force, $\delta(\phi)$ = delta function), and $\phi$ is the angle around the ring.

Equation (V-1) is valid only when the thickness of the ring is much smaller than one wavelength (This also means that the thickness has to be very small compared with the radius.) and when $\omega < \frac{c_L}{a}$, where $c_L$ is the longitudinal velocity in the ring material.

A rather straightforward method of solving Eq. (V-1) under given boundary conditions is to expand the velocity in terms of the eigenfunctions cos $n\phi$. With the use of this method the velocity at the driving point may be obtained:

$$v_o = \sum_{n=1}^{\infty} v_n = \frac{F}{i\omega m\pi a} \sum_{n=0}^{\infty} \frac{n^2}{(1+n^2) - n^2(n^2-1)^2\alpha} \qquad \text{(V-2)}$$

where $\alpha = \frac{B}{ma^4\omega^2}$ . (V-3)

For rings with rectangular cross section $\alpha = \frac{Eh^2}{12\rho a^4\omega^2}$ .

From these equations we get the so-called "modal impedance"

$$Z_{Fn} = \frac{F}{v_n} = \frac{i\omega m\pi a}{n^2}\left[1+n^2-n^2(n^2-1)^2\alpha\right] \quad . \tag{V-4}$$

Using the equation for the resonance frequencies $\omega_n$ of the ring (see Love[20])

$$\omega_n^2 \doteq \frac{B}{ma^4}\,\frac{n^2(n^2-1)^2}{n^2+1} \quad , \tag{V-5}$$

we get

$$Z_{Fn} = i\omega m\pi a\left(\frac{1+n^2}{n^2}\right)\left(1-\frac{\omega_n^2}{\omega^2}\right) \quad . \tag{V-6}$$

If the actual rather than the modal impedance is of interest, we have to carry out the summation in Eq. (V-2). For $n^2 \gg 1$ this can be done analytically. The result is

$$\frac{1}{Z_F} = \frac{\beta}{4i\omega ma}\left[\text{ctn }\pi\beta + 1\right] \tag{V-7}$$

where $\beta = (1/\alpha)^{1/4}$.

## VI. MOMENT IMPEDANCE OF BEAMS

### A. Excitation of Bending Waves

The moment impedance of beams* excited into flexural wave motion can be found by calculations which are nearly identical to those used in Section II-A. This is not surprising since the bending wave equation for the angular velocity w has the same form as Eq. (II-1).

The moment impedance of a beam extending from $x = -\infty$ to $x +\infty$ driven at $x = 0$ is

$$Z_M = \frac{2ES\kappa^2 k}{\omega}(1+i) = \frac{Z_F}{k^2} \qquad \text{(VI-1)}$$

[$Z_F$ = force impedance given by Eq. (II-4)].

For a beam extending from $x = C$ to $x = +\infty$ driven at $x = 0$, we get (under the boundary condition $\left.\frac{d^2w}{dx^2}\right|_{x=0} = 0$)

$$Z_M = \frac{Z_F}{k^2} \qquad \text{(VI-2)}$$

[$Z_F$ = force impedance given by Eq. (II-5)].

---

* For the geometries considered here see Fig. 1 and replace the force by a moment, e.g., by two opposite forces a short distance apart.

It is now very easy to find the moment impedance of beams with reflecting devices; all we have to do is to use the equation in Section II, to replace mc by $mc/k^2$, and to insert the reflection coefficients $r_w$ for angular velocities instead of the reflection coefficient r for flexural velocities. The latter substitution is no problem at all, because it turns out that in general $r_w = -r$. This means that the list in Appendix B is useful for moment impedances, too*.

Thus we can summarize: The moment impedance of a beam can be found from the force impedance of the same beam by replacing mc by $mc/k^2$ and r by -r.

B. Excitation of Torsional Waves

If a beam is excited by a moment in such a way that torsional waves are excited, the moment impedance can be found by solving the torsional wave equation

$$\frac{d^2w}{dx^2} + \frac{\omega^2}{c_T^2} w = 0 \tag{VI-3}$$

for the proper boundary conditions**. In Eq. (VI-3)

$$c_T = \sqrt{\frac{G}{\rho}} \tag{VI-4}$$

* The reflection coefficient $R_W$ for the near field could also be found in a very simple way because of the general relation $R_W = iR$.

** Strictly speaking Eq. (VI-3) is true only for cylindrical beams, but it is also a very good approximation for rectangular beams as long as the ratio of the lengths of the two sides is not too large compared to unity.

is the propagation velocity for torsional waves and $G = \frac{E}{2(1+\mu)}$ is the shear modulus.

Since Eqs. (VI-3) and (VI-4) are very similar to Eqs. (II-23) and (II-24), the moment impedances for torsional waves are also very similar to the force impedances for longitudinal waves.

For a beam extending from $x = +\infty$ to $x = -\infty$, driven somewhere in between we get

$$Z_M = 2S\kappa^2\sqrt{G\rho} \tag{VI-5}$$

($S$ = cross-section of the beam, $\kappa$ = radius of gyration).

For a beam extending from $x = 0$ to $x = \infty$, driven at $x = 0$ we get

$$Z_M = S\kappa^2\sqrt{G\rho} \tag{VI-6}$$

For beams with reflecting devices we can use Eq. (II-27)-(II-29), if we replace $S$ by $S\kappa^2$ and $E$ by $G$.

## VII. MOMENT IMPEDANCE OF PLATES

In the case of a force acting on a plate it is possible to give an impedance which is independent of the shape and size of the area over which the force is acting, provided that the area is sufficiently small. Unfortunately the same does not hold in the case of moment impedance.

If the moment is applied in form of a dipole of forces with mutual distance d, the moment impedance found by using Eq. (III-1) is, according to Cremer[21],

$$Z_M = \frac{4D}{\omega} \frac{1}{\frac{1}{4} + \frac{i}{\pi}(1-\ln 2kd)} \quad (kd \ll 1) \quad . \qquad \text{(VII-1)}$$

On the other hand for a moment applied over a disc of radius a Dyer[22] found using Eq. (III-1)

$$Z_M = \frac{4D}{\omega} \frac{1}{\frac{1}{4} - \frac{i}{\pi}\ln ka} \quad (ka \ll 1) \quad . \qquad \text{(VII-2)}$$

We see that the moment impedance is not independent of the shape and area over which it is applied; furthermore for $a \rightarrow 0$ the absolute value of the impedance would go to zero.

These difficulties still exist if the more accurate bending wave equations by Mindlin[23] are used; these equations also take into account the local shear deformations which always exist near the source. In this case Dyer[22] found for the impedance applied over a disc of radius a

$$Z_M = \frac{2D(1+L)}{\omega\left[\frac{1}{4} - \frac{i}{\pi}\ln ka + \frac{i}{\pi}\frac{2L}{1-\mu}\left(\frac{h}{\pi a}\right)^2\right]} \tag{VII-3}$$

In this equation h is the thickness of the plate. The quantities L and the third term in the denominator can be found from Fig. 7.

Although Eqs. (VII-1)-(VII-3) give very different values, they have one common feature--the real parts in the denominator are equal. This fact is of physical significance since the power transmitted by a localized moment is

$$P = \frac{1}{2}|M^2|\,\mathrm{Re}\left\{\frac{1}{Z_M}\right\} . \tag{VII-4}$$

In the first two cases this gives

$$P = \frac{|M^2|\omega}{32D} = \frac{|M|^2}{4}\,\frac{k^2}{8\sqrt{Dm}} \;; \tag{VII-5}$$

in the third case we get

$$P = \frac{M^2\omega}{16D(1+L)} \tag{VII-5a}$$

which is not very different from Eq. (VII-5).

In summary: If only the mechanical power transmitted by a localized moment is of interest, the area and shape over which the moment is applied is not of interest. In this case the power is approximately

given by Eq. (VII-5). If, however, the impedance itself is important (e.g., when it has to be added to a source impedance), it is necessary to use Eq. (VII-3).

## VIII. MOMENT IMPEDANCE OF RINGS

The moment impedance of rings can be found in a way that is very similar to that used in Section V. If a dipole of forces with moment M is acting in the radial direction, the angular velocity at the excitation point is

$$w_o = \frac{M}{i\omega m\pi a}\frac{1}{(2\pi a)^2}\sum_{n=1}^{\infty}\frac{n^4}{(1+n^2)-n^2(n^2-1)^2\alpha}, \tag{VIII-1}$$

with

$$\alpha = \frac{B}{ma^4\omega^2} .$$

From Eq. (VIII-1) we find for the modal moment impedance

$$Z_{Mn} = i\omega m\pi a\left(\frac{1+n^2}{n^2}\right)\left(1-\frac{\omega_n^2}{\omega^2}\right)\left(\frac{2\pi a}{n}\right)^2 \tag{VIII-2}$$

($\omega_n$ see Eq. V-5).

The asymptotic value of $1/Z_M$ for $n^2 >> 1$ turns out to be

$$\frac{1}{Z_M} = \frac{\beta^3}{16\pi^2 i\omega m a^3}(\mathrm{ctn}\,\pi\beta + 1) . \tag{VIII-3}$$

where $\beta = (1/\alpha)^{1/4}$

# APPENDIX A

## LIST OF REFLECTION COEFFICIENTS FOR BENDING WAVES ON BEAMS*

### 1. Change in Cross-Section or Material

If $m_1$, $B_1$, $k_1$ denote the mass, bending stiffness, and wave number before the discontinuity, and $m_2$, $B_2$, $k_2$ the same parameters behind it, the reflection coefficient r for the wave field is given by (see Cremer[5/])

$$r = \frac{2\alpha(1-\beta^2) - i\beta(1-\alpha)^2}{\beta(1+\alpha)^2 + 2\alpha(1+\beta^2)} \quad . \tag{A1}$$

For the near-field reflection coefficient Cremer obtains

$$R = \frac{\beta(1-\alpha^2) - i\beta(1-\alpha^2)}{\beta(1+\alpha)^2 + 2\alpha(1+\beta^2)} \quad , \tag{A2}$$

where

$$\alpha = \sqrt{\frac{m_2 B_2}{m_1 B_1}} \text{ and } \beta = \frac{k_2}{k_1} \quad .$$

### 2. Impeding Masses

If $m_o$, $\theta$, K denote the mass, rotatory inertia and the radius of gyration of the added mass, and m, k the surface mass and bending wave number of the beam, the reflection coefficients are given by (see Cremer[5/])

---

* For sketches of the geometries see Fig. 8.

$$r = \frac{-\mu+\epsilon^2\mu^3+\frac{1}{2}\epsilon^2\mu^4}{(\mu+\epsilon^2\mu^3)-i(4+\mu-\epsilon^2\mu^3-\frac{1}{2}\epsilon^2\mu^4)} , \tag{A3}$$

and

$$R = \frac{-\epsilon^2\mu^3-\frac{1}{2}\epsilon^2\mu^4+i(\mu-\frac{1}{2}\epsilon^2\mu^4)}{(\mu+\epsilon^2\mu^3)-i(4+\mu-\epsilon^2\mu^3-\frac{1}{2}\epsilon^2\mu^4)} . \tag{A4}$$

Here the abbreviations $\epsilon = \frac{m}{m_o} K$ and $\mu = \frac{m_o}{m} k$ are introduced.

For frequencies more than one octave above the "Sperrfrequenz" which is given by $kK = 1$ the above equations can be approximately written as

$$1-|r|^2 \approx \left(\frac{2m}{m_o k}\right)^2 ; \; R \approx -(1-i) . \tag{A5}$$

3. Corners and Other Junctions

a) Vibrations in the Plane Formed by the Beams.

This case was investigated by Cremer[5] taking into consideration the excitation of longitudinal waves. The corresponding formulas can be found in Cremer's paper. For many practical cases, however, it is not necessary to use these rather complicated formulas, since the errors are not too great in the region where the simple bending wave theory is valid [see Eq. (II-1)]. Therefore, for most cases the approximate formulas

$$r = -\frac{1+iq}{1+q} \; ; \quad R = -q\frac{1-i}{1+q} \tag{A6}$$

are sufficient.

Here q is given by the bending stiffness B and the wave number k of the primary beam and the same quantities $B_\nu$ and $k_\nu$ of the adjoined beams

$$q = \frac{1}{Bk} \sum_{\nu=1}^{3} B_\nu k_\nu \quad . \tag{A7}$$

b. Vibrations Perpendicular to the Plane Formed by the Beams

If this case is treated accurately, the excitation of torsional waves must be considered. This leads to the rather complicated expressions given in Ref. 11. If the excitation of torsional waves is neglected the reflection coefficients are

$$r = \frac{1-iq}{1+q} \quad , \tag{A8}$$

and

$$R = q \, \frac{i-1}{1+q} \quad . \tag{A9}$$

The quantity q is given by Eq. (A7) but since the bending stiffness is in general different along different axes the numerical value of q might be very different.

Equations (A8) and (A9) are not very good approximations, and are especially poor at low frequencies. They can serve only as very rough estimates.

4. Terminations

| | | |
|---|---|---|
| For clamped ends: | $r = -i;\ R = -(1-i)$ . | (A10) |
| For simply supported ends: | $r = -1;\ R = 0$ . | (A11) |
| For free ends: | $r = -i;\ R = 1-i$ . | (A12) |

If it is possible to describe the termination by a force impedance $\frac{F}{v} = Z_F$ and a moment impedance $\frac{M}{w} = Z_M$, the reflection coefficients are given by

$$r = -\frac{(1+i)(1+\tau\nu)-2(\tau+i\nu)}{(1-i)(1+\tau\nu)-2(\tau-i\nu)} \tag{A13}$$

and

$$R = \frac{-2i(1-\tau\nu)}{(1-i)(1+\tau\nu)-2(\tau-i\nu)} , \tag{A14}$$

where $\nu = -\frac{i\omega}{Bk} Z_M$ and $\tau = \frac{i\omega}{Bk^3} Z_F$ .

The reflection coefficients for free, clamped and supported beams can easily be verified from these equations. Also, the case of a beam attached to a plate or to a large mass is included in Eq. (A13) and (A14). It should be mentioned that $|r|$ is always unity when $\nu$ and $\tau$ are real i.e., when $Z_F$ and $Z_M$ are purely imaginary.

## APPENDIX B

### IMPEDANCE OF BEAMS WITH REFLECTING DEVICES

A model of a beam with two reflecting devices is shown in Fig. 1e. To derive the velocity at the driving point we assume that L and $\ell$ are at least larger than half a bending wavelength; this implies that the near fields excited at the discontinuities can be neglected.

The velocity at the driving point x = 0 can be calculated by noting that the primary waves excited at x = 0 propagate to x = $\ell$ or x = -L. At these points they are partially reflected and propagate in the opposite direction until they encounter the next discontinuity where they are reflected once more, etc. The addition of all these partial waves gives the actual velocity.

The velocity of the primary waves is given by Cremer[5] as

$$v_+ = \frac{F}{4cm} (e^{-ikx} - ie^{-kx}) \text{ for } x \geqslant 0$$

and (B1)

$$v_- = \frac{F}{4cm} (e^{+ikx} - ie^{kx}) \text{ for } x \leqslant 0$$

If we first consider the $v_+$ wave only, we find (neglecting near fields) that after the first reflection the wave has the form

$$v_{R1} = \frac{F}{4cm} r_\ell \, e^{-ik(2\ell - x)} \tag{B2}$$

($r_\ell$ = reflection coefficient at x = $\ell$).

If the wave described by Eq. (B2) hits the discontinuity at $x = -L$, another reflected wave is generated, namely

$$v_{R2} = \frac{F}{4cm} r_\ell r_L \, e^{-ik(2\ell+2L+x)} \tag{B3}$$

The next reflections at $x = \ell$ and $x = -L$ give

$$v_{R3} = \frac{F}{4cm} r_L \, e^{-ik(2\ell-x)} \, r_L r_\ell \, e^{-2ik(L+\ell)} \tag{B4}$$

and

$$v_{R4} = \frac{F}{4cm} r_\ell r_L \, e^{-ik(2\ell+2L+x)} \, r_L r_\ell \, e^{-2ik(L+\ell)} \tag{B5}$$

and so on. It can be seen immediately that the summation of all of the reflections forms a geometrical series in $r_L r_\ell \, e^{-2ik(L+\ell)}$. The summation therefore can be carried out easily, and if we do this for $v_+$ and $v_-$ we obtain

$$v_o = \frac{F}{4cm} \frac{1-i+r_L \, e^{-2ikL}+r_\ell \, e^{-2ik\ell}+(1+i)r_\ell r_L \, e^{-2ik(\ell+L)}}{1-r_\ell r_L \, e^{-2ik(\ell+L)}} \tag{B6}$$

($v_o$ = velocity at $x = 0$).

## APPENDIX C

### FORCE IMPEDANCE OF A FINITE PLATE BEAM SYSTEM

The input impedance of an infinite bar which is attached over its entire length to an infinite plate was derived by Lamb[19]. An extension of this problem will be treated, namely, the force impedance of a finite beam attached to a plate. The geometry of the system can be seen in Fig. 6b. To simplify the calculations it is assumed that the plate and the beam are simply supported at the lines $y = 0$ and $y = \ell$, i.e., the velocities and the moments at these lines are zero.

We assume the plate and the beam to be thin, so that the vibrations of the plate are given by

$$\Delta\Delta v_P - k_P^4 v_P = 0 \tag{C1}$$

and the vibrations of the beam by

$$\frac{d^4 v_B}{dy^4} - k_B^4 v_B = \frac{i\omega}{B} p_B \quad . \tag{C2}$$

In these and the following equations the subscript B corresponds to the beam and P to the plate: $\Delta$ is the Laplacian operator, $k_B$, $k_P$ the bending wave numbers, B the bending stiffness of the beam, and D the bending rigidity of the plate.* $p_B$ is the pressure acting on

---

* See note on p. 22.

the beam and consists of two parts, the exciting pressure $p_{BE}$ given by a point force F applied at a point $y_o$, and a pressure $p_{BR}$ which includes the backward reaction from the plate.

Because of the simple boundary conditions, the velocity of the beam can be expressed as

$$v_B = \sum_{n=1}^{\infty} v_{Bn} \sin \frac{n\pi y}{\ell} \quad . \tag{C3}$$

To find the unknown coefficients $v_{Bn}$, we must know the pressures $p_B$ and $p_{BR}$. To this end we expand the velocity of the plate in a series similar to Eq. (C3). (This can be done because of the similarity of the boundary conditions.) We write

$$v_P = \sum_{n=1}^{\infty} f_n(x) \sin \frac{n\pi y}{\ell} \quad . \tag{C4}$$

The functions $f_n(x)$ describe the propagation of bending waves into the plate. They can be found by inserting Eq. (C4) into Eq. (C1) giving

$$\frac{d^4}{dx^4} f_n(x) - 2 \frac{n^2\pi^2}{\ell^2} \frac{d^2}{dx^2} f_n(x) + \left[ \left(\frac{n\pi}{\ell}\right)^4 - k_P^4 \right] f_n(x) = 0 \quad . \tag{C5}$$

This equation can easily be solved by introducing the exponential function. Thus we get

$$f_n(x) = v_{Pn-}e^{i\rho x}+v'_{Pn-}e^{\sigma x}+v_{Pn+}e^{-i\rho x}+v'_{Pn+}e^{-\sigma x} \quad , \tag{C6}$$

where

$$\rho = \sqrt{k_P^2 - \frac{n^2\pi^2}{\ell^2}} \text{ and } \sigma = \sqrt{k_P^2 + \frac{n^2\pi^2}{\ell^2}} \quad . \tag{C7}$$

For our model we must consider that no waves can come from infinity. Therefore, we have two different solutions for $x < 0$, and $x > 0$.

$$v_{P+} = \sum_{n=1}^{\infty} \left(v_{Pn+}e^{-i\rho x}+v'_{Pn+}e^{-\sigma x}\right) \sin \frac{n\pi}{\ell} y \text{ for } x > 0$$

and (C8)

$$v_{P-} = \sum_{n=1}^{\infty} \left(v_{Pn-}e^{i\rho x}+v'_{Pn-}e^{\sigma x}\right) \sin \frac{n\pi}{\ell} y \text{ for } x < 0 \quad .$$

Because of the symmetry of the problem we can put $v_{Pn+} = v_{Pn-}$ and $v'_{Pn+} = v'_{Pn-}$. Furthermore, it is sufficient to consider one of the equations (C8) and to double the reacting pressure obtained in this way.

The coefficients $v_{Pn+}$ and $v'_{Pn+}$ can be expressed in terms of $v_{Bn}$ by using the following boundary conditions.

$$v_B = v_{P+}; \text{ or } v_{Bn} = v_{Pn+} \text{ for } x = 0 \tag{C9}$$

and

$$\frac{\partial v_{P+}}{\partial x} = 0 \text{ for } x = 0 \quad . \tag{C10}$$

The first of these equations expresses the fact that the velocity of the beam and the plate must be equal at the line $x = 0$. The second equation means that the angular velocity at $x = 0$ must be zero, because there are no bending moments in the x direction.

Equation (C10), together with Eq. (C8) and Eq. (C9), gives

$$v_{Pn+} = \frac{v_{Bn}}{2k_P^2} (\sigma^2 + i\rho\sigma) \tag{C11}$$

and

$$v'_{Pn+} = \frac{v_{Bn}}{2k_P^2} (\rho^2 - i\rho\sigma) \tag{C12}$$

Now we are able to express the pressure $p_{BR}$ in terms of $v_{Bn}$. The pressure is given by

$$p_{BR} = \frac{2D}{i\omega} \left[ \frac{\partial \Delta v_{P+}}{\partial x} \right]_{\ell = 0} \quad . \tag{C13}$$

Equations (C8), (C11), (C12), and (C13) yield, after some algebra,

$$p_{BR} = \frac{2D}{i\omega} \sum_{n=1}^{\infty} v_{Bn}(\rho^2\sigma - i\rho\sigma^2) \sin \frac{n\pi}{\ell} y \quad . \tag{C14}$$

Now we can expand the exciting pressure $p_{BE}$ in the same series by writing

$$p_{BE} = \sum_{n=1}^{\infty} p_{BEn} \sin \frac{n\pi}{\ell} y \quad , \tag{C15}$$

where $p_{BEn}$ can easily be computed in the special case of a point force

$$p_{BE} = F\delta(y-y_o) \tag{C16}$$

which has the Fourier coefficients

$$p_{BEn} = \frac{2}{\ell} F \sin \frac{n\pi y_o}{\ell} \tag{C17}$$

By inserting Eqs. (C13), (C14), (C15), and (C17) in Eq. (C2) we get

$$\left[\left(\frac{n\pi}{\ell}\right)^4 - k_B^4\right] v_{Bn} = \frac{2}{\ell} \frac{i\omega}{B} F \sin \frac{n\pi y_o}{\ell} + 2 \frac{D}{B}\left(\sigma\rho^2 - i\rho\sigma^2\right) v_{Bn} \quad . \tag{C18}$$

This expression gives us the quantities $v_{Bn}$. We get finally

$$v_B = \frac{2i\omega F}{B\ell} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi y_o}{\ell} \sin \frac{n\pi}{\ell} y}{\left(\frac{n\pi}{\ell}\right)^4 - k_B^4 - 2i \frac{D}{B}\left(\frac{n^4\pi^4}{\ell^4} - k_P^4\right)\left(\frac{1}{\sqrt{k_P^2 - \frac{n^2\pi^2}{\ell^2}}} + \frac{i}{\sqrt{k_P^2 + \frac{n^2\pi^2}{\ell^2}}}\right)} \tag{C19}$$

This equation is very similar to that derived by Lamb[19]; the main difference is that a sum instead of an integral is obtained for the finite case.

To obtain some further information from Eq. (C19) let us investigate two special cases:. first, the case of a very light and flexible beam on a stiff plate; second, the case of a very stiff beam attached on a rather flexible plate;

Case A

In this case, especially when there is no beam, i.e., B = 0, we get, for the velocity at the driving point

$$v_{BO} \approx + \frac{\omega F}{2k_P^2 D} \sum_{n=1}^{\infty} \sin^2 \frac{n\pi y_o}{\ell} \left[ \frac{1}{\sqrt{k_P^2 \ell^2 - n^2\pi^2}} - i \frac{1}{\sqrt{k_P^2 \ell^2 + n^2\pi^2}} \right] . \quad \text{(C20)}$$

Equation (C20) is identical with Eq. (III-9), as expected.

Case B

In this case, which occurs very often in practice (e.g., when a thin plate is stiffened by a stringer), it is reasonable to compare the vibrations of the beam-plate system with the vibrations of a damped beam.

In the case of a point-driven damped beam we insert the complex value

$$\underline{k}_B^4 = k_B^4 (1-i\eta) \qquad \text{(C21)}$$

in Eq. (C2). The vibrations can be calculated as shown above. The only difference is that there is no reacting pressure $p_{BR}$. Thus we get for the velocity of a point-driven damped beam

$$\underline{v}_B = \frac{2i\omega F}{B\ell} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi y_o}{\ell}}{\left(\frac{n\pi}{\ell}\right)^4 - k_B^4 + i\eta k_B^4} \sin \frac{n\pi y}{\ell} \quad . \tag{C22}$$

To compare this equation with Eq. (C19) we take into consideration that for $D \ll Bk_P$ the value of the sum in Eq. (C19) is mainly determined by the behavior of the denominator in the vicinity of $n\pi/\ell = k_B$. Therefore, we make no great error if we put $n\pi/\ell = k_B$ in the last term of the denominator of Eq. (C19). So we get

$$\underline{v}_B \approx \frac{2i\omega F}{B\ell} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi y_o}{\ell} \sin \frac{n\pi y}{\ell}}{\left(\frac{n\pi}{\ell}\right)^4 - k_B^4 + 2i \frac{D}{B} \left(k_P^4 - k_B^4\right)\left(\frac{1}{\sqrt{k_P^2 - k_B^2}} + \frac{i}{\sqrt{k_P^2 + k_B^2}}\right)} \quad . \tag{C23}$$

If we compare this result with Eq. (C22), we see that the coupling of a flexible plate to a stiff beam has approximately the same effect as damping the beam with a loss factor

$$\eta_R \approx 2 \frac{D}{B} \frac{\left(\frac{k_P^4}{k_B^4} - 1\right)}{\sqrt{k_P^2 - k_B^2}} \tag{C24}$$

and changing the bending wave number because of an "additional mass" of magnitude

$$\frac{\Delta m_B}{m_B} \approx 2\,\frac{D}{B}\,\frac{\left(\dfrac{k_P^4}{k_B^4} - 1\right)}{\sqrt{k_P^2 - k_B^2}} \quad . \tag{C25}$$

These two equations can be simplified a little more if we set $k_P^4 >> k_B^4$, which is at least correct if beam and plate are made out of the same material. Hence

$$\eta_R \approx \frac{2}{k_P}\,\frac{m_P}{m_B} \quad ; \quad \Delta m_B \approx \frac{2}{k_P}\,mp \quad . \tag{C26}$$

## REFERENCES

1. Muster, D. and Plunkett, R., Chapter 18 in Noise Reduction Ed. by L. Beranek, McGraw Hill Book Company, 1960.

2. Lowe, R. T., and Crede L. E., NOISE Control 3, November 1957, p. 21.

3. Skudrzyk, E. J., J. Acoust. Soc. Am. 31, (1960), p. 68.

4. Kuhl, W. and Schröder, F. K., Acustica (Beihefte) 6 (1956) p. 79.

5. Cremer, L., The Propagation of Structure-borne Sound, Dept. of Sci. and Ind. Res., London, Sponsored Research. Report N.1 Series B.

6. Kurtze, G., Tamm, W. and Vogel, S., Acustica 5 (1955) p. 223

7. Morse, P.W., Vibration and Sound, McGraw Hill Book Company, 1948.

8. Zener, C., Phys. Rev. 59 (1942), p. 669.

9. Sneddon, I. N., Proc. Cambr. Phil. Soc. 41 (1945) p. 27.

10. Cremer, H. u L., Frequenz 2 (1948) p. 61.

11. Heckl, M., Wave Propagation on Grills, BBN Report in preparation.

12. Huber, M. T., Z. öster. Ing. Arch. Ver. 30 (1914) p. 557.

13. Hoppmann, W. H., Proc. Soc. ~~Expt~~l. Stress Anal. 14 (1956) p. 137.

14. Girkmann, K., Flächentragwerke, 3. Auflage Springer Verlag, Wien 1954.

15. Timoshenko, S. P. and Woinowsky-Krieger, S., Theory of Plates and Shells, McGraw Hill Book Company (1959) Chapter 11.

16. Heckl, M., Acustica <u>10</u> (1960) p. 109.

17. Jahnke, E. and Emde, F., Tables of Functions, Dover Publication 1945.

18. Skudrzyk, E. J., J. Acoust. Soc. Am. <u>30</u>, (1958) p. 1140.

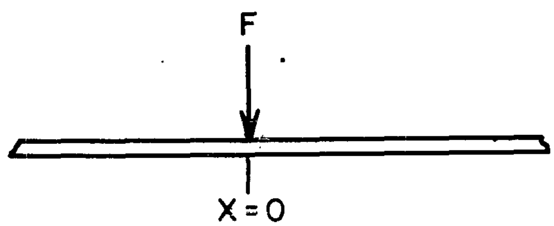
19. Lamb, G. L., Jr., Bolt Beranek and Newman Inc., Report No. 603, ONR Contract Nonr 2322(00), December 1958.

20. Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity, Dover, New York, p. 452.

21. Cremer, L., Acustica <u>4</u> (1954) p. 273.

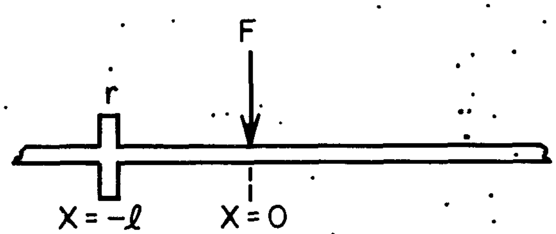
22. Dyer, I., J. Acoust. Soc. Am. <u>32</u> (1960) p. 1290 see also BBN Report No. 727.

23. Mindlin, R. D., J. Appl. Mech. <u>18</u> (1951) p. 31.

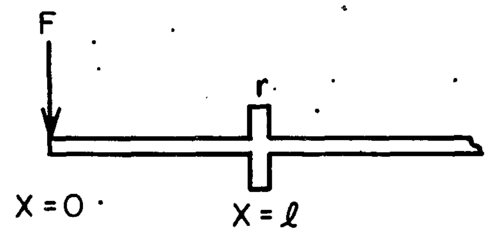



(a) eq. (II,4)

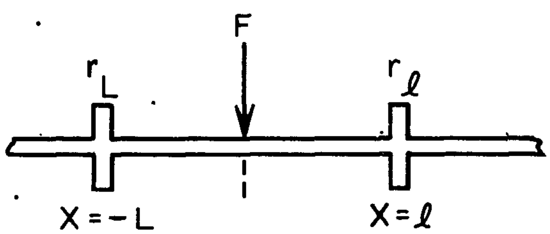

(b) eq. (II,5)




(c) eq. (II,6)




(d) eq. (II,7)

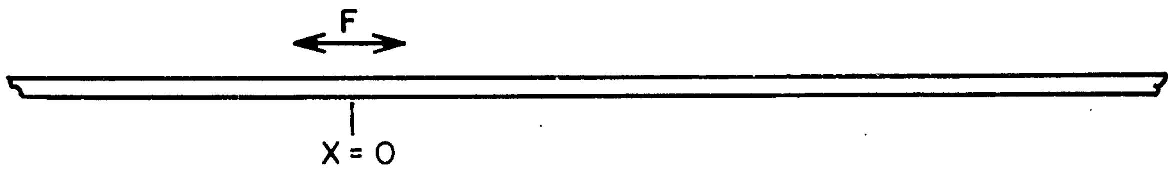



(e) eq. (II, 10 AND 11)

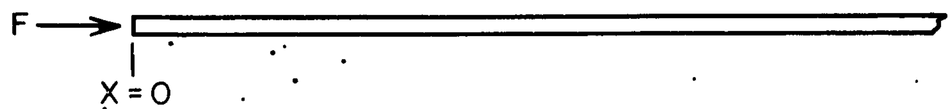
NOTE:

TO FIND THE MOMENT IMPEDANCE USE THE CORRESPONDING EQUATION FOR FORCE IMPEDANCE AND REPLACE cm BY $cm/k^2$ AND r BY −r.

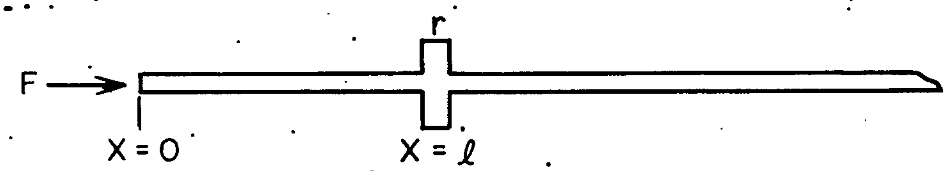
FIG. 1 LIST OF BEAM IMPEDANCE (BENDING)



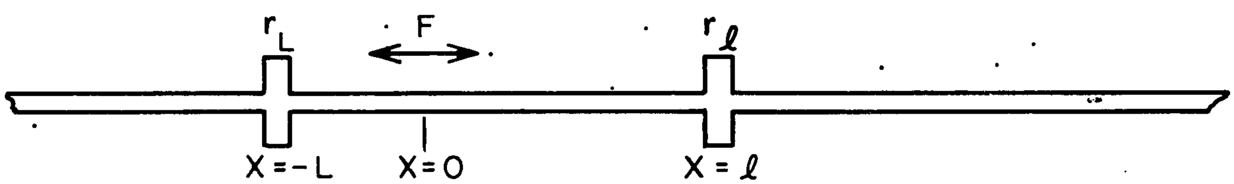

(a) eq. (II, 25)




(b) eq. (II, 26)

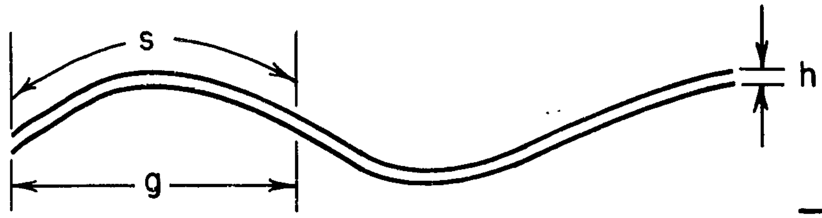



(c) eq. (II, 27)

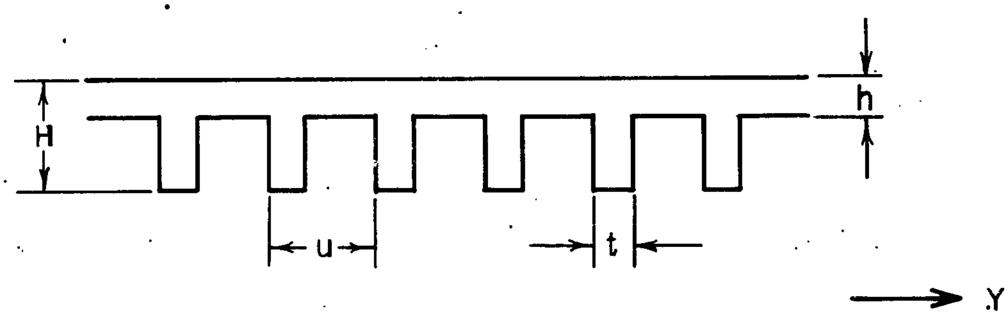



(d) eq. (II, 29)

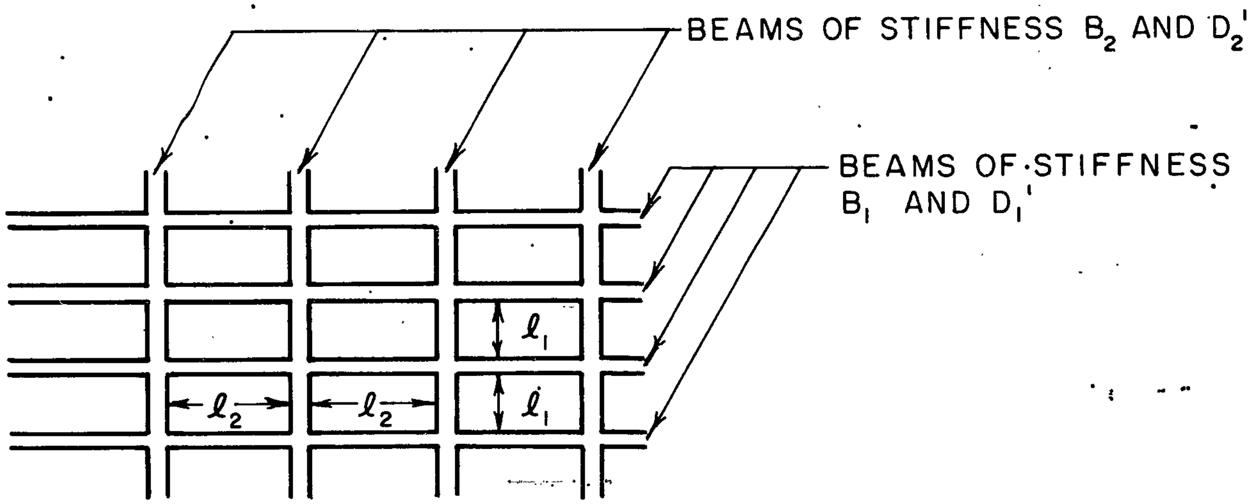
FIG. 2 LIST OF BEAM IMPEDANCES (LONGITUDINAL)




(a) CORRUGATED PLATE




(b) PLATE WITH RIBS




(c) GRILLS

FIG. 3 DIMENSIONS OF ORTHOTROPIC PLATES (IMPEDANCES eq. III,4 – III,7)

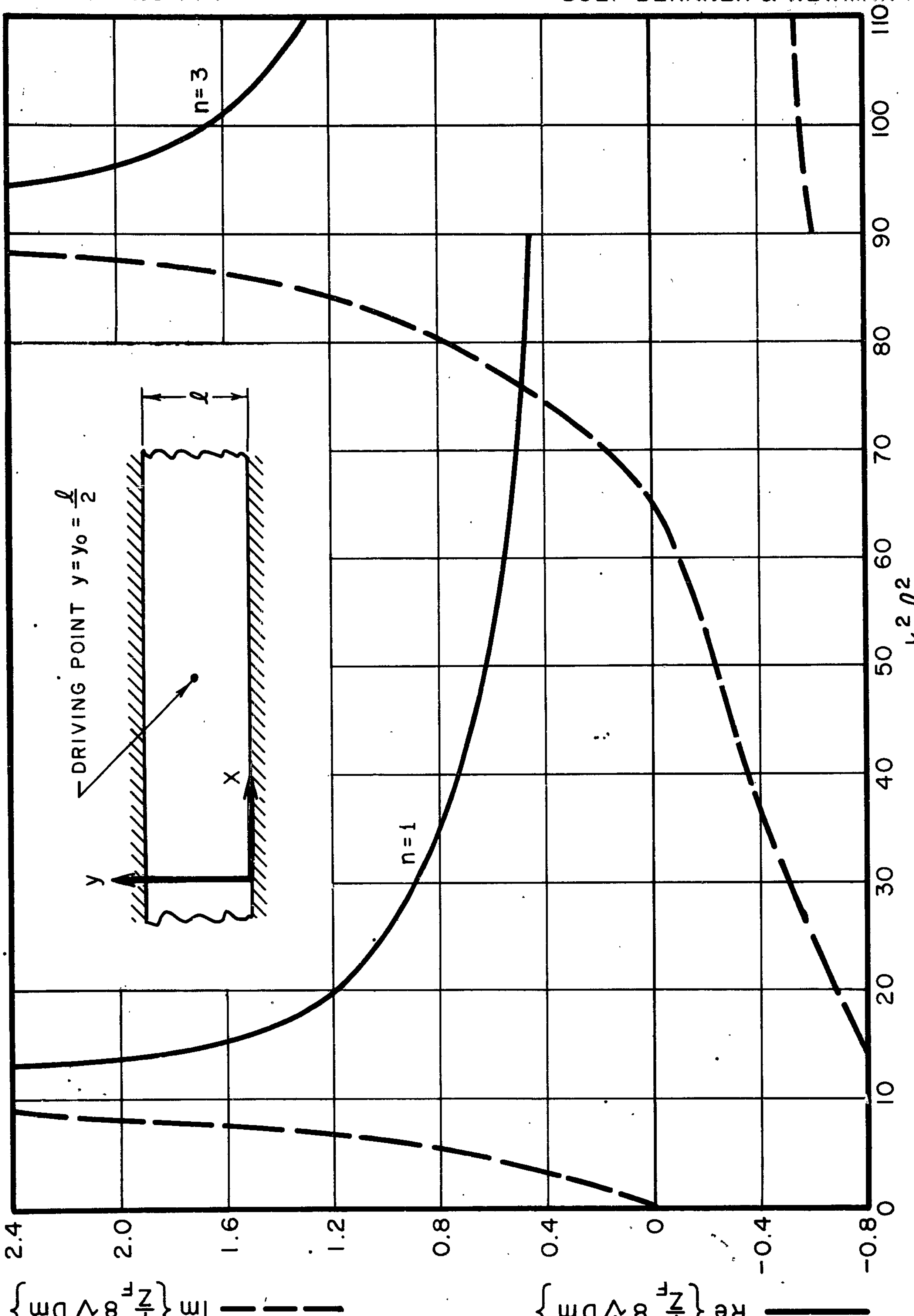



FIG.4 FORCE ADMITTANCE OF A PLATE INFINITE IN ONE DIRECTION, SIMPLE SUPPORTED IN THE OTHER ONE. (DRIVING POINT IN THE MIDDLE)

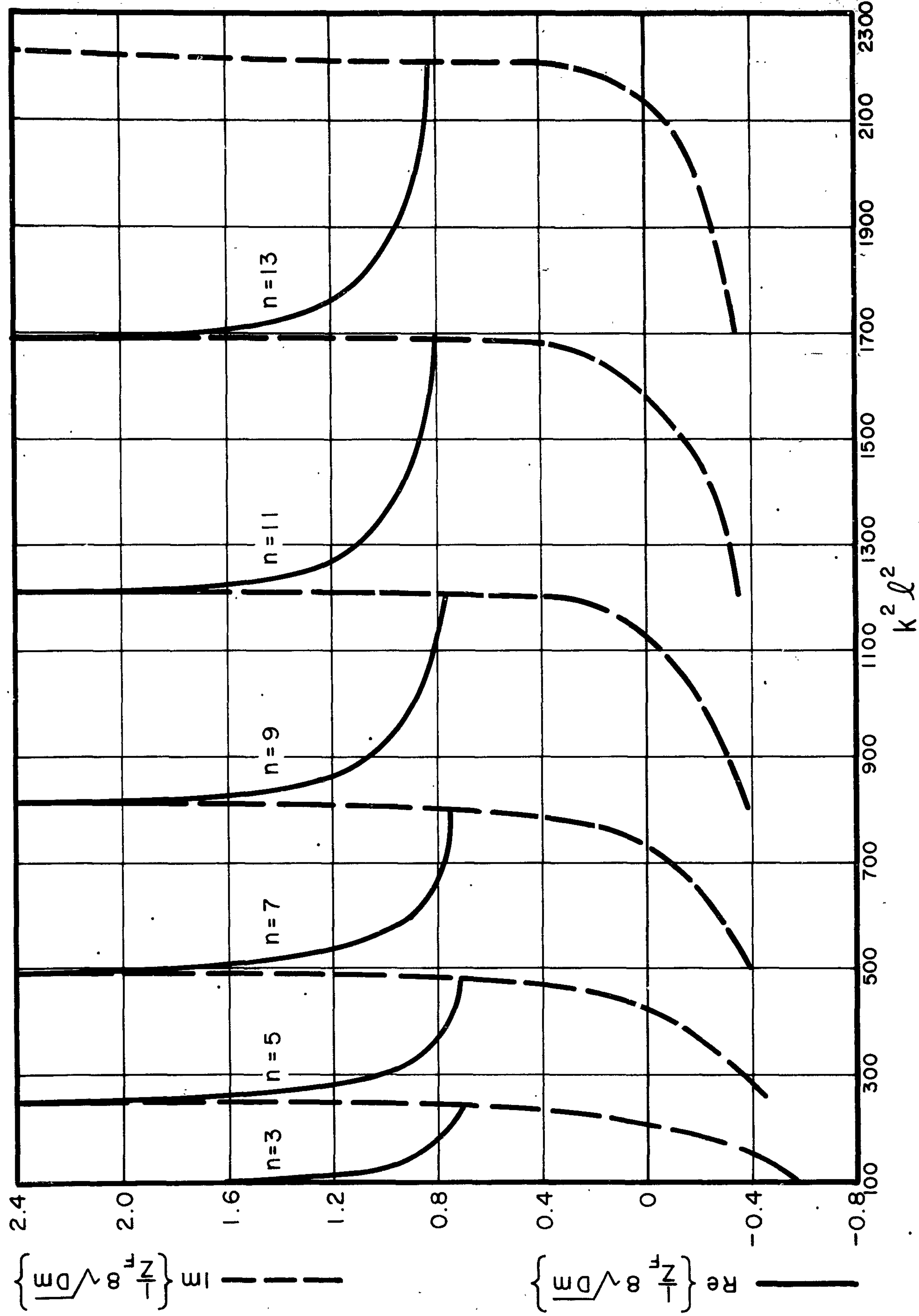
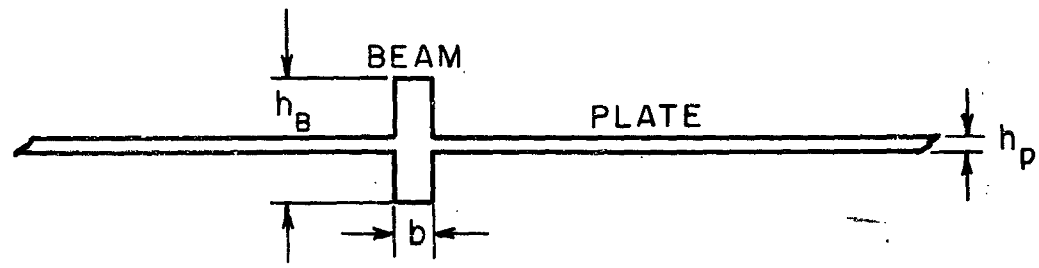
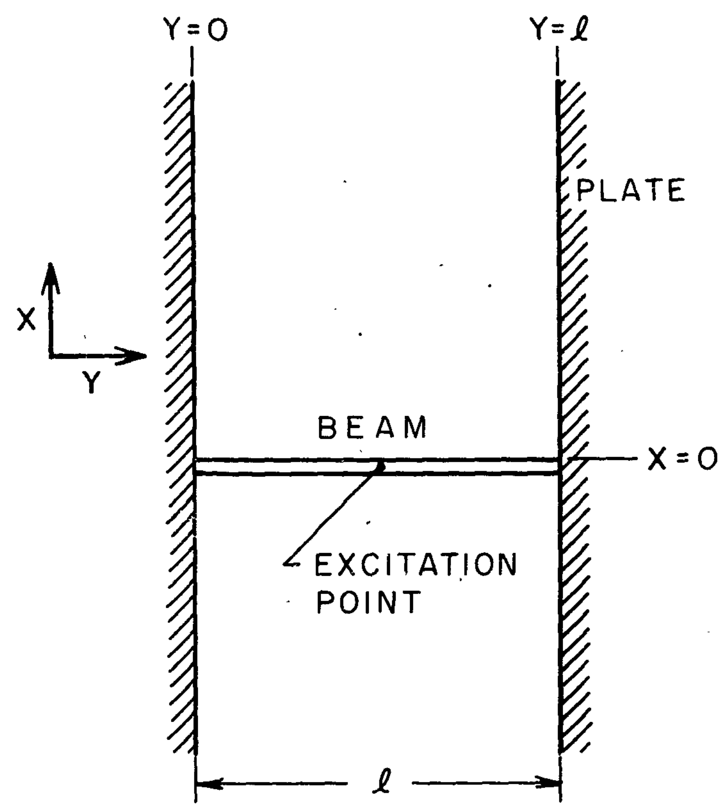



FIG.5 FORCE ADMITTANCE OF A PLATE INFINITE IN ONE DIRECTION, SIMPLE SUPPORTED IN THE OTHER ONE. (DRIVING POINT IN THE MIDDLE)




(a) INFINITE CASE: IMPEDANCE, SEE SECTION IV A




CROSS-SECTION, SEE UPPER SKETCH

(b) FINITE SYSTEM

FIG. 6 SKETCH OF BEAM PLATE SYSTEM

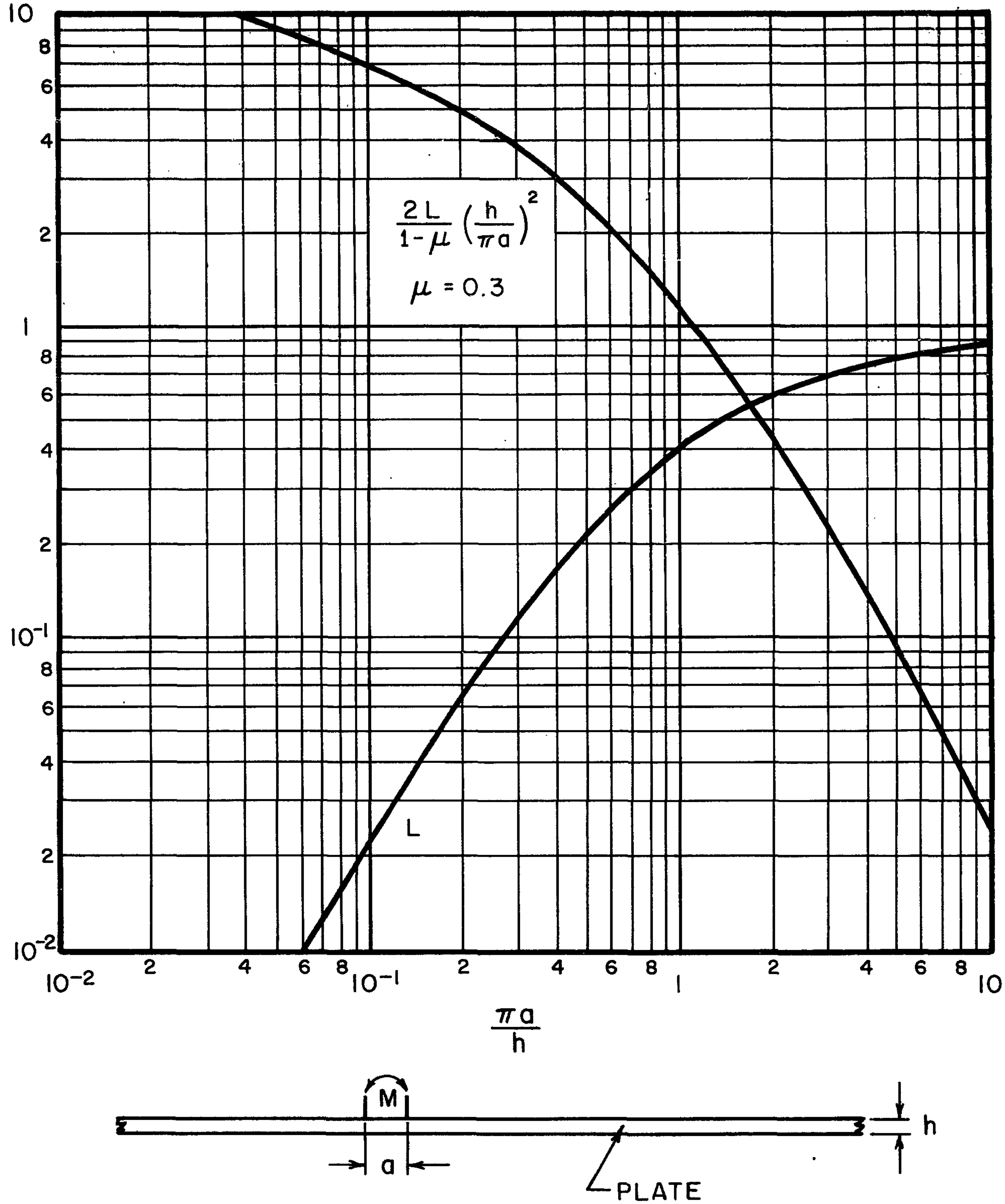



FIG. 7 QUANTITIES FOR THE CALCULATION OF THE THIN PLATE MOMENT IMPEDANCE

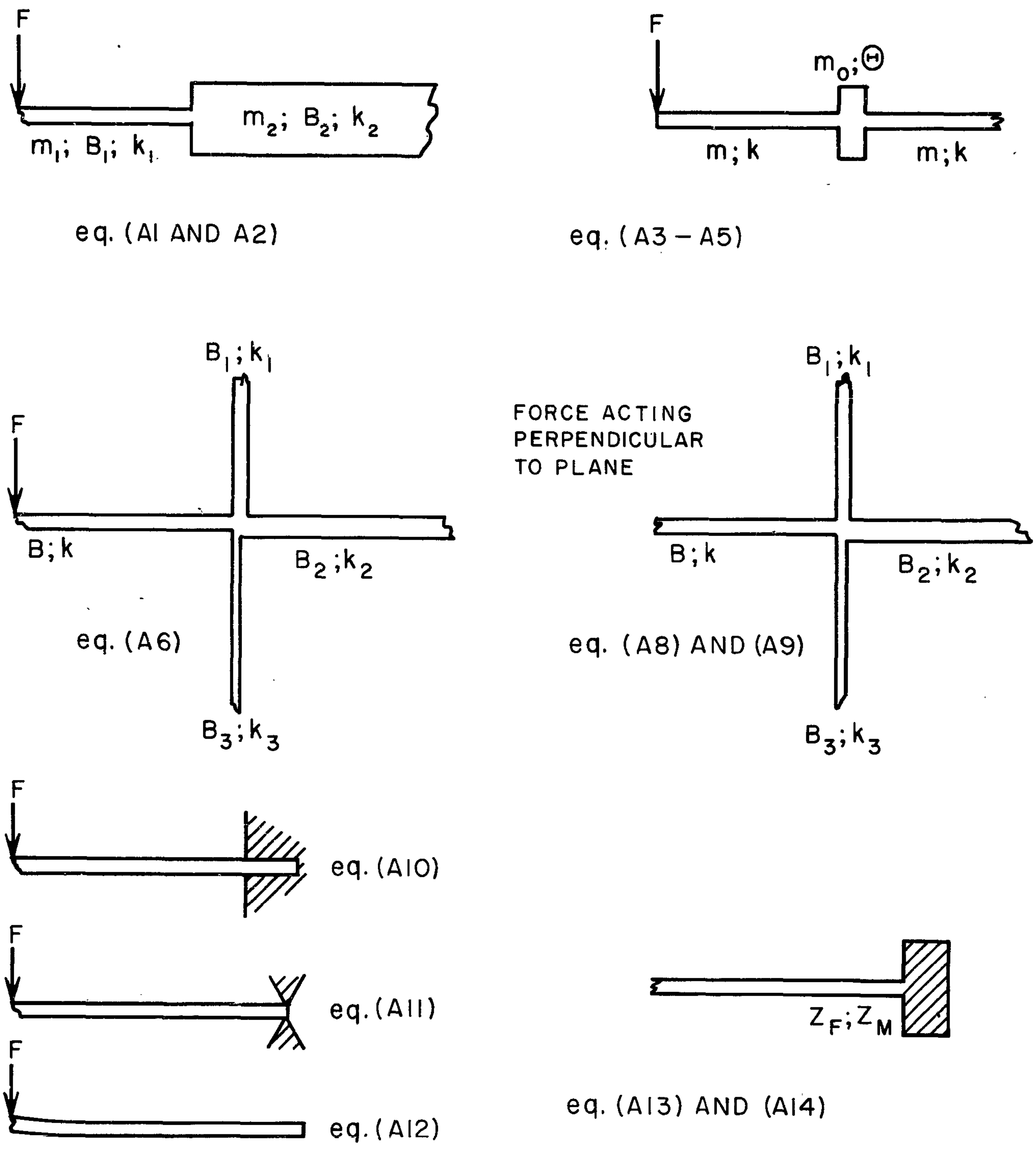



FIG. 8 LIST OF REFLECTION COEFFICIENTS

DISTRIBUTION LIST

Contract Nonr 2322(00)

(Single Copy unless otherwise indicated)

Office of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 411 (6 copies)

Office of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 438

Office of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 439

Director
U. S. Naval Research Laboratory
Sound Division
Washington 25, D. C.

Commanding Officer
Office of Naval Research
Box 39, Navy No. 100
FPO, New York (5 copies)

Office of Technical Services
Department of Commerce
Washington, D. C.

Armed Services Technical Information Agency
Arlington Hall Station
Arlington 12, Virginia (10 copies)

Commander
U. S. Naval Ordnance Laboratory
Acoustics Division
White Oak
Silver Spring, Maryland

Director
U. S. Navy Underwater Sound Reference Laboratory
Office of Naval Research
P. O. Box 3629
Orlando, Florida

Commanding Officer and Director
U. S. Navy Underwater Sound Lab.
Fort Trumbull
New London, Connecticut
Attn: Dr. W. A. Von Winkle

Commander
U. S. Naval Air Development Center
Johnsville, Pennsylvania

Director
National Bureau of Standards
Connecticut Ave. and Van Ness St. N. W.
Washington 25, D. C.
Attn: Chief of Sound Section

Office of Chief Signal Officer
Department of the Army
Pentagon Building
Washington 25, D. C.

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C.
Attn: Dr. M. Strasberg

Commanding Officer and Director
U. S. Naval Engineering Experiment Station
Annapolis, Maryland

Superintendent
U. S. Navy Postgraduate School
Monterey, California
Attn: Prof. L. E. Kinser

Commanding Officer
Air Force Cambridge Research Center
230 Albany Street
Cambridge 39, Massachusetts

Chief
Office of Ordnance Research
Box C. M., Duke Station
Durham, N. C.

Commanding Officer and Director
U. S. Navy Electronics Laboratory
San Diego 52, California

National Science Foundation
1520 H Street N. W.
Washington, D. C.

Commanding Office
U. S. Navy Mine Defense Laboratory
Panama City, Florida

U. S. Naval Academy
Annapolis, Maryland
Attn: Library

Chief, Physics Division
Office of Scientific Research
Air Research + Development Command
Washington 25, D. C.

Harvard University
Acoustics Laboratory
Division of Applied Science
Cambridge 38, Massachusetts
Attn: Prof. F. V. Hunt

Brown University
Department of Physics
Providence 12, R. I.
Attn: Prof. P. J. Westervelt (1 copy)
Prof. R. B. Lindsay (1 copy)

University of California
Department of Physics
Los Angeles, California
Attn: Prof. R. W. Leonard

University of California
Marine Physical Laboratory of
the Scripps Institution
of Oceanography
San Diego 52, California
Attn: Dr. F. M. Spiess

Chief Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: 335 (Tech. Library) (1 copy)
345 (2 copies)
420 (1 copy)
440 (1 copy)

Director
Columbia University
Hudson Laboratories
145 Palisades Street
Dobbs Ferry, New York

Woods Hole, Oceanographic
Institution
Woods Hole, Massachusetts

Ordnance Research Laboratory
Pennsylvania State University
University Park, Pennsylvania
Attn: Dr. E. Skudrzyk (1 copy)
Dr. J. Johnson (1 copy)

Massachusetts Institute of
Technology
Cambridge 39, Massachusetts
Attn: Physics Dept.
Prof. P. M. Morse (1 copy)
Prof. K. U. Ingard (1 copy)

Columbia University
New York 27, New York
Attn: Prof. F. L. DiMaggio (1 copy)
Prof. C. M. Harris (1 copy)

University of Minnesota
Department of Electrical Engineering
Minneapolis 14, Minnesota
Attn: Prof. R. F. Lambert

Cambridge Acoustical Associates
1278 Mass. Avenue
Cambridge 38, Massachusetts
Attn: Dr. M. C. Junger

J. G. Engineering Research Associates
3709 Callaway Avenue
Baltimore 15, Maryland
Attn: Dr. J. E. Greenspan

Stromberg Carlson
Rochester, New York
Attn: Dr. J. V. Bouyoucos

Chesapeake Instrument Corp.
Shadyside, Maryland
Attn: Mr. J. W. Fitzgerald

Defense Research Laboratory
University of Texas
Austin 12, Texas
Attn: Mr. C. M. McKinney